

# Model of human collective decision-making in complex environments

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## Abstract

A continuous-time Markov process is proposed to analyze how a group of humans solves a complex task, consisting in the search of the optimal set of decisions on a fitness landscape. Individuals change their opinions driven by two different forces: (i) the self-interest, which pushes them to increase their own fitness values, and (ii) the social interactions, which push individuals to reduce the diversity of their opinions in order to reach consensus. Results show that the performance of the group is strongly affected by the strength of social interactions and by the level of knowledge of the individuals. Increasing the strength of social interactions improves the performance of the team. However, too strong social interactions slow down the search of the optimal solution and worsen the performance of the group. In particular, we find that the threshold value of the social interaction strength, which leads to the emergence of a superior intelligence of the group, is just the critical threshold at which the consensus among the members sets in. We also prove that a moderate level of knowledge is already enough to guarantee high performance of the group in making decisions.

Keywords: Decision making, social interactions, complexity, Markov chains.

## I. INTRODUCTION

The ability of groups to solve complex problems that exceed individual skills is widely recognized in natural, human, and artificial contexts. Animals in groups, e.g. flocks of birds, ant colonies, and schools of fish, exhibit collective intelligence when performing different tasks as which direction to travel in, foraging, and defence from predators [1, 2]. Artificial systems such as groups of robots behaving in a self organized manner show superior performance in solving their tasks, when they adopt algorithms inspired by the animal behaviors in groups [3–6]. Human groups such as organizational teams outperform the single individuals in a variety of tasks, including problem solving, innovative projects, and production issues [7–11].

The superior ability of groups in solving tasks originates from collective decision making: agents (animals, robots, humans) make choices, pursuing their individual goals (forage, survive, etc.) on the basis of their own knowledge and amount of information (position, sight, etc.), and adapting their behavior to the actions of the other agents. The group-living enables social interactions to take place as a mechanism for knowledge and information sharing [9, 12–21]. Even though the single agents possess a limited knowledge, and the actions they perform usually are very simple, the collective behavior, enabled by the social interactions, leads to the emergence of a superior intelligence of the group. This property is known as swarm intelligence [22–24] and wisdom of crowds [25].

In this paper we focus on human groups solving complex combinatorial problems. Many managerial problems including new product development, organizational design, and business strategy planning may be conceived as problems where the effective combinations of multiple

and interdependent decision variables should be identified [26–29]. We develop a model of collective decision making, which attempts to capture the main drivers of the individual behaviors in groups, i.e., self-interest and consensus seeking. We consider that individuals make choices based on rational calculation and self-interested motivations. Agent's choices are made by optimizing the perceived fitness value, which is an estimation of the real fitness value based on the level of agent's knowledge [1, 30, 31]. However, any decision made by an individual is influenced by the relationships he/she has with the other group members. This social influence pushes the individual to modify the choice he/she made, for the natural tendency of humans to seek consensus and avoid conflict with people they interact with [32].

We use the Ising-Glauber dynamics [33, 34] to model the social interactions among group members. The  $N-K$  model [35, 36] is employed to build the fitness landscape associated with the problem to solve. A continuous-time Markov chain governs the decision-making process, whose complexity is controlled by the parameter  $K$ . We define the transition rate of individual's opinion change as the product of the Ising-Glauber rate ([34]), which implements the consensus seeking [38–41], and an exponential rate [42, 43], which speeds up or slows down the change of opinion, to model the rational behavior of the individual.

Herein, we explore how both the strength of social interactions and the level of knowledge of the members influence the group performance. We identify in which circumstances human groups are particularly effective in solving complex problems. We extend previous studies highlighting the efficacy of collective decision making in presence of a noisy environment [44], and in conditions of cognitive limitations [2, 9, 45–47]. This decision-

making model might be proposed as optimization technique belonging to the class of swarm intelligence techniques [22, 48–52].

## II. THE MODEL

We consider a human group made of  $M$  socially interacting members, which is assigned to solve a complex task. The task consists in solving a combinatorial decision making problem by identifying the set of decisions (choice configuration) with the highest fitness. The fitness function is built employing the  $N-K$  model [35–37]. A  $N$ -dimensional vector space of decisions is considered, where each choice configuration is represented by a vector  $\mathbf{d} = (d_1, d_2, \dots, d_N)$ . Each decision is a binary variable that may take only two values  $+1$  or  $-1$ , i.e.  $d_i = \pm 1$ ,  $i = 1, 2, \dots, N$ . The total number of decision vectors is therefore  $2^N$ . Each vector  $\mathbf{d}$  is associated with a certain fitness value  $V(\mathbf{d})$  computed as the weighted sum of  $N$  stochastic contributions  $W_j(d_j, d_1^j, d_2^j, \dots, d_K^j)$ , each decision leads to total fitness depending on the value of the decision  $d_j$  itself and the values of other  $K$  decisions  $d_i^j$ ,  $i = 1, 2, \dots, K$ . Following the classical  $N-K$  [35–37] procedure (more details are provided in Appendix A), the quantities  $W_j \in [0, 1]$  are determined as randomly generated  $2^{K+1}$ -element “interaction tables”. The fitness function of the group is defined as

$$V(\mathbf{d}) = \frac{1}{N} \sum_{j=1}^N W_j(d_j, d_1^j, d_2^j, \dots, d_K^j) \quad (1)$$

The integer index  $K = 0, 1, 2, \dots, N-1$  is the number of interacting decision variables, and tunes the complexity of the problem. The complexity of the problem increases with  $K$ . Note that, for  $K > 2$ , in computational complexity theory, finding the optimum of the fitness function  $V(\mathbf{d})$  is classified as a NP-complete decision problem [37]. This makes this approach particularly suited in our case.

We model the level of knowledge of the  $k$ -th member of the group (with  $k = 1, 2, \dots, M$ ) by defining the  $M \times N$  competence matrix  $\mathbf{D}$ , whose elements  $D_{kj}$  take the value  $D_{kj} = 1$  if the member  $k$  knows the contribution of the decision  $j$  to the total fitness  $V$ , otherwise  $D_{kj} = 0$ . Based on the level of knowledge each member  $k$  computes his/her own perceived fitness (self-interest) as

$$V_k(\mathbf{d}) = \frac{\sum_{j=1}^N D_{kj} W_j(d_j, d_1^j, d_2^j, \dots, d_K^j)}{\sum_{j=1}^N D_{kj}}. \quad (2)$$

Each member of the group makes his/her choices driven by the rational behavior, which pushes him/her to increase the self-interest, and by social interactions, which push the member to seek consensus within the group. When  $D_{kj} = 0$ , for  $j = 1, 2, \dots, N$  the  $k$ -th member possesses no knowledge about the fitness function, and his

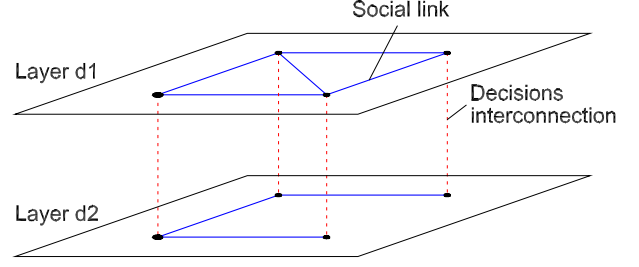


FIG. 1: A two-layer multiplex network. Each layer is associated with one single decision variable. Blue lines represent social links between members, whereas dashed red lines represent the interconnections between the decisions of each member.

choices are driven only by consensus seeking. Note that the choice configuration that optimizes the perceived fitness Eq. (2), does not necessarily optimize the group fitness Eq. (1). This makes the mechanism of social interactions, by means of which knowledge is transferred, crucial for achieving high-performing decision-making process. We build the matrix  $\mathbf{D}$ , by randomly choosing  $D_{kj} = 1$  with probability  $p \in [0, 1]$ , and  $D_{kj} = 0$  with probability  $1 - p$ . By increasing  $p$  from 0 to 1 we control the level of knowledge of the members, which affects the ability of the group in maximizing the fitness function Eq. (1).

All members of the group make choices on each of the  $N$  decision variables  $d_j$ . Therefore, the state of the  $k$ -th member ( $k = 1, 2, \dots, M$ ) is identified by the  $N$ -dimensional vector  $\sigma_k = (\sigma_k^1, \sigma_k^2, \dots, \sigma_k^N)$ , where  $\sigma_k^j = \pm 1$  is a binary variable representing the opinion of the  $k$ -th member on the  $j$ -th decision. For any given decision variable  $d_j$ , individuals  $k$  and  $h$  agree if  $\sigma_k^j = \sigma_h^j$ , otherwise they disagree. Within the framework of Ising’s approach [38, 40, 41], disagreement is characterized by a certain level of conflict  $E_{kh}^j$  (energy level) between the two socially interacting members  $k$  and  $h$ , i.e.  $E_{kh}^j = -J\sigma_k^j\sigma_h^j$ , where  $J$  is the strength of the social interaction. Therefore, the total level of conflict on the decision  $d_j$  is given by:

$$E^j = - \sum_{(k,h)} J\sigma_k^j\sigma_h^j \quad (3)$$

where the symbol  $()$  indicates that the sum is limited to the nearest neighbors, i.e. to those individuals which are directly connected by a social link.

A multiplex network [53–57] with  $N$  different layers is defined. On each layer, individuals share their opinions on a certain decision variable  $d_j$  leading to a certain level of conflict  $E^j$ . The graph of social network on the layer  $d_j$  is described in terms of the symmetric adjacency matrix  $\mathbf{A}^j$  with elements  $A_{kh}^j$ . The interconnections between different layers represent the interactions among the opinions of the same individual  $k$  on the decision variables. Figure 1 shows an example of a mul-

tiplex network with only two layers, where the dashed lines connecting the different decision layers represent the interaction between the opinions that each member has on the decision variables. This interaction occurs via the  $N - K$  perceived fitness, i.e. changing the opinion on the decision variable  $j$  causes a modification of the perceived pay-off, which also depends on the opinions the member has on the remaining decision variables. In order to model the dynamics of decision-making in terms of a continuous-time Markov process, we define the state vector  $\mathbf{s}$  of the entire group as  $\mathbf{s} = (s_1, s_2, \dots, s_n) = (\sigma_1^1, \sigma_1^2, \dots, \sigma_1^N, \sigma_2^1, \sigma_2^2, \dots, \sigma_2^N, \dots, \sigma_M^1, \sigma_M^2, \dots, \sigma_M^N)$  of size  $n = M \times N$ , and the block diagonal adjacency matrix  $\mathbf{A} = \text{diag}(\mathbf{A}^1, \mathbf{A}^2, \dots, \mathbf{A}^N)$ . For any given  $l$ -th component  $s_l = \sigma_k^j$  of the vector  $\mathbf{s}$  it is possible to uniquely identify the member  $k$  and the decision variable  $j$  by means of the relations  $k = \text{quotient}(l - 1, M) + 1$ , and  $j = \text{mod}(l - 1, M) + 1$ . The total level of conflict can be then rephrased as

$$E(\mathbf{s}) = -\frac{1}{2}J\mathbf{A}\mathbf{s} \cdot \mathbf{s} = -\frac{1}{2}J \sum_{ij} A_{ij}s_i s_j \quad (4)$$

Observe that  $A_{ii} = 0$  (with  $i = 1, \dots, n$ ). In Eq. (4) the term  $1/2$  avoids that each couple of members  $k$  and  $h$  be double counted. Now let be  $P(\mathbf{s}, t)$  the probability that, at time  $t$ , the state vector takes the value  $\mathbf{s}$  out of  $2^n$  possible states. The time evolution of the probability  $P(\mathbf{s}, t)$  obeys the master equation

$$\begin{aligned} \frac{dP(\mathbf{s}, t)}{dt} = & - \sum_l w(\mathbf{s}_l \rightarrow \mathbf{s}'_l) P(\mathbf{s}_l, t) \\ & + \sum_l w(\mathbf{s}'_l \rightarrow \mathbf{s}_l) P(\mathbf{s}'_l, t) \end{aligned} \quad (5)$$

where  $\mathbf{s}_l = (s_1, s_2, \dots, s_l, \dots, s_n)$ ,  $\mathbf{s}'_l = (s_1, s_2, \dots, -s_l, \dots, s_n)$ . The transition rate  $w(\mathbf{s}_l \rightarrow \mathbf{s}'_l)$  is the probability per unit time that the opinion  $s_l$  flips to  $-s_l$  while the others remain temporarily fixed. Recalling that flipping of opinions is governed by social interactions and self-interest a possible ansatz for the transition rates is

$$\begin{aligned} w(\mathbf{s}_l \rightarrow \mathbf{s}'_l) = & \frac{1}{2} \left[ 1 - s_l \tanh \left( \beta J \sum_h A_{lh} s_h \right) \right] \\ & \times \exp \{ \beta' [\Delta V(\mathbf{s}'_l, \mathbf{s}_l)] \} \end{aligned} \quad (6)$$

In Eq. (6) the pay-off function  $\Delta V(\mathbf{s}'_l, \mathbf{s}_l) = \bar{V}(\mathbf{s}'_l) - \bar{V}(\mathbf{s}_l)$ , where  $\bar{V}(\mathbf{s}_l) = V_k(\sigma_k)$ , is simply the change of the fitness perceived by the agent  $k = \text{quotient}(l - 1, M) + 1$ , when its opinion  $s_l = \sigma_k^j$  on the decision  $j = \text{mod}(l - 1, M) + 1$  changes from  $s_l = \sigma_k^j$  to  $s'_l = -\sigma_k^j$ . The transition rates  $w(\mathbf{s}_l \rightarrow \mathbf{s}'_l)$  have been chosen to be the product of the transition rate of the Ising-Glauber dynamics [34] (see also Appendix B), and the Weidlich exponential rate  $\exp \{ \beta' [\Delta V(\mathbf{s}'_k, \mathbf{s}_k)] \}$  [42, 43]. Note that Eq. (6) satisfies the detailed balance condition (see Appendix C). In Eq. (6) the quantity  $\beta$  is the inverse of

the so-called social temperature and is a measure of the chaotic circumstances, which lead to a random opinion change. The term  $\beta'$  is related to the degree of uncertainty associated with the information about the perceived fitness (the higher  $\beta'$  the less the uncertainty).

To solve the Markov process Eqs. (5, 6), we employ a simplified version of the exact stochastic simulation algorithm proposed by Gillespie [58, 59]. A brief summary of the algorithm is provided in Appendix D. The algorithm allows to generate a statistically correct trajectory of the stochastic process Eqs. (5, 6).

### III. MEASURING THE PERFORMANCE OF THE COLLECTIVE DECISION-MAKING PROCESS

The group fitness value Eq. (1) and the level of agreement between the members (i.e. social consensus) are used to measure the performance of the collective-decision making process. To calculate the group fitness value, the vector  $\mathbf{d} = (d_1, d_2, \dots, d_N)$  needs to be determined. To this end, consider the set of opinions  $(\sigma_1^j, \sigma_2^j, \dots, \sigma_M^j)$  that the members of the group have about the decision  $j$ , at time  $t$ . The decision  $d_j$  is obtained by employing the majority rule, i.e. we set

$$d_j = \text{sgn} \left( M^{-1} \sum_k \sigma_k^j \right), \quad j = 1, 2, \dots, N \quad (7)$$

If  $M$  is even and in the case of a parity condition,  $d_j$  is, instead, uniformly chosen at random between the two possible values  $\pm 1$ . The group fitness is then calculated as  $V[\mathbf{d}(t)]$  and the ensemble average  $\langle V(t) \rangle$  is then evaluated. The efficacy of the group in optimizing  $\langle V(t) \rangle$  is then calculated in terms of normalized average fitness  $\langle V(t) \rangle / V_{\max}$  where  $V_{\max} = \max[V(\mathbf{d})]$ .

The consensus of the members on the decision variable  $j$  is measured as follows. We define the average opinion  $\bar{\sigma}^j$  of the group on the decision  $j$

$$\bar{\sigma}^j = \frac{1}{M} \sum_k \sigma_k^j \quad (8)$$

Note that the quantity  $\bar{\sigma}^j$  ranges in the interval  $-1 \leq \bar{\sigma}^j \leq 1$ , and that  $\bar{\sigma}^j = \pm 1$  only when full consensus is reached. Therefore, a possible measure of the consensus among the members on the decision variable  $j$  is given by the ensemble average of the time-dependent quantity  $C^j = (\bar{\sigma}^j)^2 \in [0, 1]$ , i.e.,

$$\langle C^j(t) \rangle = \frac{1}{M^2} \sum_{kh} \langle \sigma_k^j(t) \sigma_h^j(t) \rangle = \frac{1}{M^2} \sum_{kh} R_{hk}^j(t) \quad (9)$$

Note that  $\langle \sigma_k^j(t) \sigma_h^j(t) \rangle = R_{hk}^j(t)$  is the correlation function of the opinions of the members  $k$  and  $h$  on the same decision variable  $j$ . Given this, a possible ansatz to

measure the entire consensus of the group on the whole set of decisions is

$$\langle C(t) \rangle = \frac{1}{N} \sum_j \langle C^j(t) \rangle = \frac{1}{M^2 N} \sum_{j=1}^N \sum_{kh=1}^M R_{hk}^j(t) \quad (10)$$

Note that  $0 \leq \langle C(t) \rangle \leq 1$ .

#### IV. SIMULATION AND RESULTS

We consider, unless differently specified, a group of  $M = 6$  members which have to make  $N = 12$  decisions. For the sake of simplicity, the network of social interactions on each decision layer  $j$  is described by a complete graph, where each member is connected to all the others. We also set  $\beta' = 10$ , since we assume that the information about the perceived fitness function is characterized by a low level of uncertainty. We simulate many diverse scenarios to investigate the influence of the parameter  $p$ , i.e. of the level of knowledge of the members, and the effect of the parameter  $\beta J$  on the final outcome of the decision-making process. The simulation is stopped at steady-state. This condition is identified by simply taking the time-average of consensus and pay-off over consecutive time intervals of fixed length  $T$  and by checking that the difference between two consecutive averages is sufficiently small. For any given  $p$  and  $\beta J$ , each stochastic process Eqs. (5, 6) is simulated by generating 100 different realizations (trajectories). For each single realization, the competence matrix  $\mathbf{D}$  is set, and the initial state of the system is obtained by drawing from a uniform probability distribution, afterwards the time evolution of the state vector is calculated with the stochastic simulation algorithm (see Appendix D). Fig. 2 shows the time-evolution of normalized average fitness  $\langle V(t) \rangle / V_{\max}$ , for  $p = 0.5$  (i.e. for a moderate level of knowledge of the members), different values of the complexity parameter  $K = 1, 5, 11$ , and different values of  $\beta J = 0.0, 0.5, 1.0$ . We observe that for  $\beta J = 0$ , i.e. in absence of social interactions [see Fig. 2(a)] the decision-making process is strongly inefficient, as witnessed by the very low value of the average fitness of the group. Each individual of the group makes his/her choices in order to optimize the perceived fitness, but, because of the absence of social interactions, he/she behaves independently from the others and does not receive any feedback about the actions of the other group members. Hence, individuals remain close to their local optima, group fitness cannot be optimized [see Fig. 2(a)], and the consensus is low [see Fig. 3(a)]. As the strength of social interactions increases, i.e.,  $\beta J = 0.5$  [Fig. 2(b)], members can exchange information about their choices. Social interactions push the individuals to seek consensus with the member who is experiencing higher payoff. In fact, on the average, those members, which find a higher increase of their perceived fitness, change opinion much faster than the others. Thus, the other members, in process of seeking consensus, skip the local optima of their

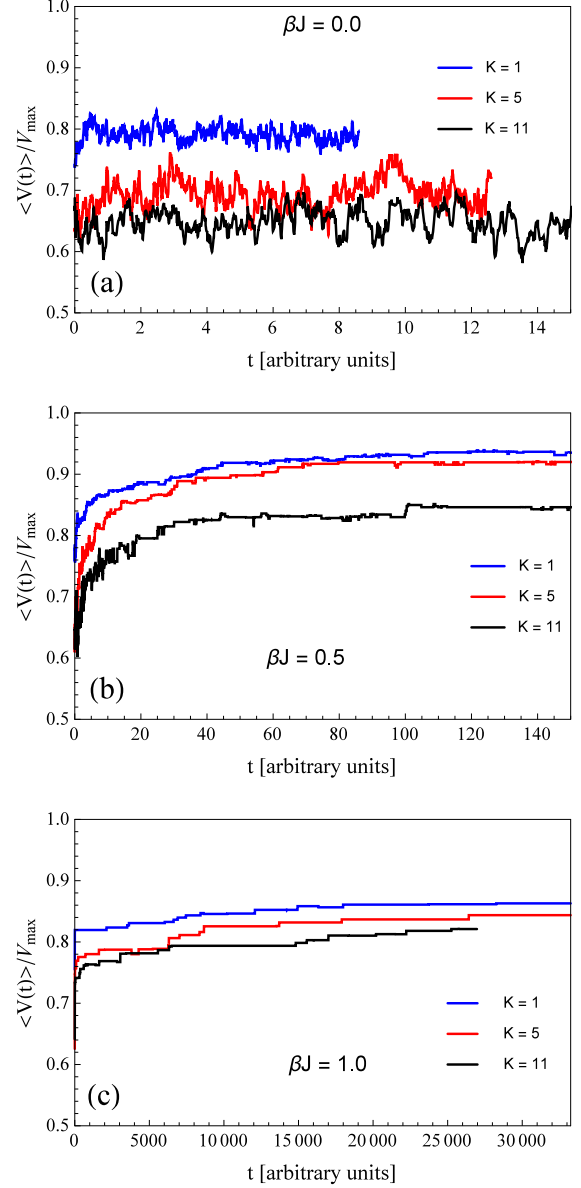


FIG. 2: The time-evolution of the normalized average group fitness  $\langle V(t) \rangle / V_{\max}$ , for  $p = 0.5$ ,  $K = 1, 5, 11$ .  $\beta J = 0.0$ , (a);  $\beta J = 0.5$ , (b);  $\beta J = 1.0$ , (c).

perceived fitness and keep exploring the landscape, leading to a substantial increase of the group performance both in terms of group fitness values [Fig. 2(b)] as well as in terms of final consensus [Fig. 3(b)]. Thus, the system collectively shows a higher level of knowledge and higher ability in making good choices than the single members (i.e., a higher degree of intelligence). It is noteworthy that when the strength of social interactions is too large,  $\beta J = 1$ , [Fig. 2(c)] the performance of the group in terms of fitness value worsens. In fact, very high values of  $\beta J$ , accelerating the achievement of consensus among the

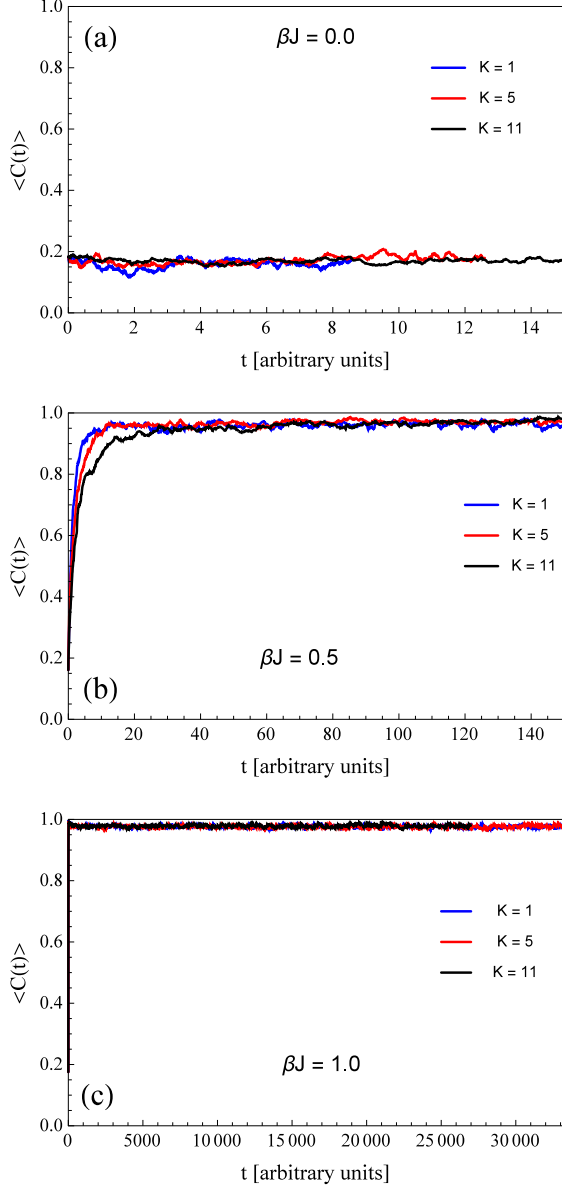


FIG. 3: The time-evolution of the statistically averaged consensus  $\langle C(t) \rangle$  for  $p = 0.5$ ,  $K = 1, 5, 11$ .  $\beta J = 0.0$ , (a);  $\beta J = 0.5$ , (b);  $\beta J = 1.0$ , (c).

members [Fig. 3(c)], significantly impede the exploration of the fitness landscape and hamper that change of opinions can be guided by payoff improvements. The search of the optimum on the fitness landscape is slowed down, and the performance of the collective decision-making decreases both in terms of the time required to reach the steady-state as well as in terms of group fitness.

Figure 2 shows that rising the complexity of the landscape, i.e. increasing  $K$ , negatively affects the performance of the collective decision-making process, but does not qualitatively change the behavior of the system.

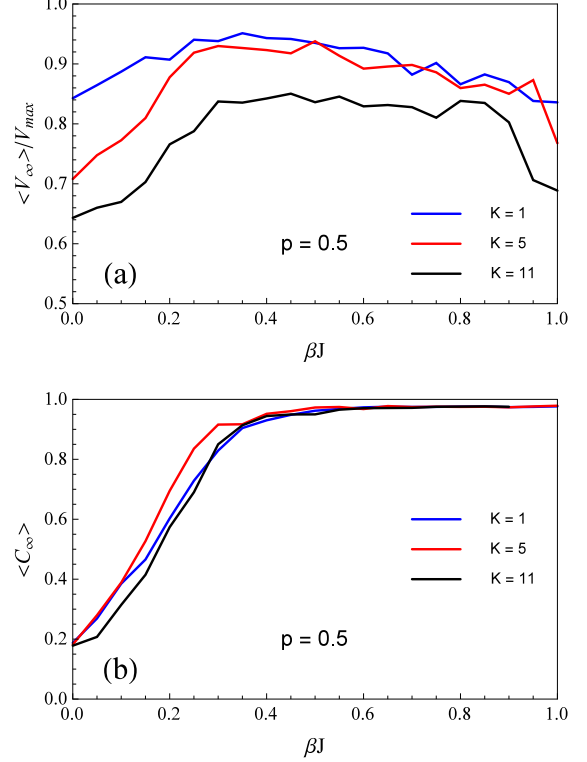


FIG. 4: The stationary values of the normalized averaged fitness  $\langle V_\infty \rangle / V_{\max}$  as a function of  $\beta J$ , (a); and of the statistically averaged consensus  $\langle C_\infty \rangle$  as a function of  $\beta J$ , (b). Results are presented for  $p = 0.5$ ,  $K = 1, 5, 11$ .

However, Figure 2(b) also shows that, in order to cause a significant worsening of the group fitness,  $K$  must take very large values, i.e.,  $K = 11$ . Instead, at moderate, but still significant, values of complexity (see results for  $K = 5$ ) the decision-making process is still very effective, leading to final group fitness values comparable to those obtained at the lowest level of complexity, i.e., at  $K = 1$ .

In Figure 3 the ensemble average  $\langle C(t) \rangle$  of the consensus among the members is shown as a function of time  $t$ , for  $p = 0.5$ ,  $K = 1, 5, 11$ , and for different values of  $\beta J = 0.0, 0.5, 1.0$ . At  $\beta J = 0$ , the consensus is low. In this case, at each time  $t$ , members' opinions are random variables almost uniformly distributed between the two states  $\pm 1$ . Hence, the quantity  $\langle C(t) \rangle$  can be analytically calculated as  $\langle C(t) \rangle \approx 1/M$ . For  $M = 6$  this gives  $\langle C(t) \rangle \approx 0.16$ , which is just the average value observed in Fig. 3(a). As the strength of social interactions rises, members more easily converge toward a common opinion. However, the random nature of the opinion dynamics still prevents full agreement from being achieved, see Fig. 3(b). This, as observed in Figure 2(b), has a very beneficial effect as individuals continue exploring the fitness landscape looking for maxima, thus leading to higher performance of the collective decision-making pro-

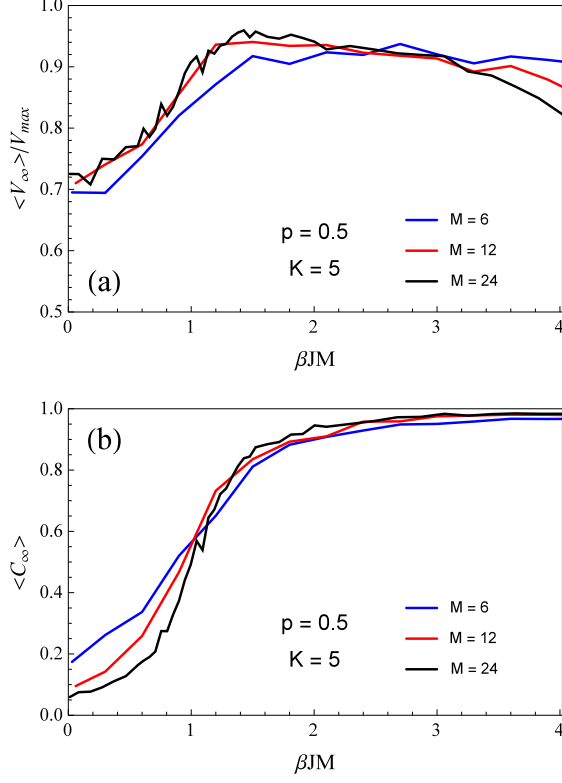


FIG. 5: The stationary values of the normalized averaged fitness  $\langle V_\infty \rangle / V_{\max}$  as a function of  $M\beta J$ , (a); and of the statistically averaged consensus  $\langle C_\infty \rangle$  as a function of  $M\beta J$ , (b). Results are presented for  $p = 0.5$ ,  $K = 5$  and for three different team sizes:  $M = 6, 12, 24$ .

cess. However, when the strength of social interactions is significantly increased, a very high value of consensus among members is rapidly achieved [see Fig. 3(c)], the exploration of the landscape is slowed down, and the performance of the decision making-process significantly worsen, see Figure 2(c). We, then, expect that, given  $\beta'$  and  $K$ , a optimum of  $\beta J$  exists, which maximizes the steady-state fitness of the group. This is, indeed, confirmed by the analysis shown in Figure 4, where the steady-state values of the normalized group fitness  $\langle V_\infty \rangle / V_{\max} = \langle V(t \rightarrow \infty) \rangle / V_{\max}$  [Fig. 4(a)], and social consensus  $\langle C_\infty \rangle = \langle C(t \rightarrow \infty) \rangle$  [Fig. 4(b)] are plotted as a function of  $\beta J$ , for  $p = 0.5$  and the three considered values of  $K = 1, 5, 11$ . results in Figure 4(a) stresses that the fitness landscape complexity (i.e., the parameter  $K$ ) marginally affects the performance of the decision-making process in terms of group fitness, provided that  $K$  does not take too high values. In fact curves calculated for  $K = 1, 5$  run close to each-other. More interesting, Figure 4 shows that increasing  $\beta J$  from zero, makes both  $\langle V_\infty \rangle / V_{\max}$  and  $\langle C_\infty \rangle$  rapidly increase. This increment is, then, followed by a region of a slow change of  $\langle V_\infty \rangle / V_{\max}$  and  $\langle C_\infty \rangle$ . It is worth noticing, that the

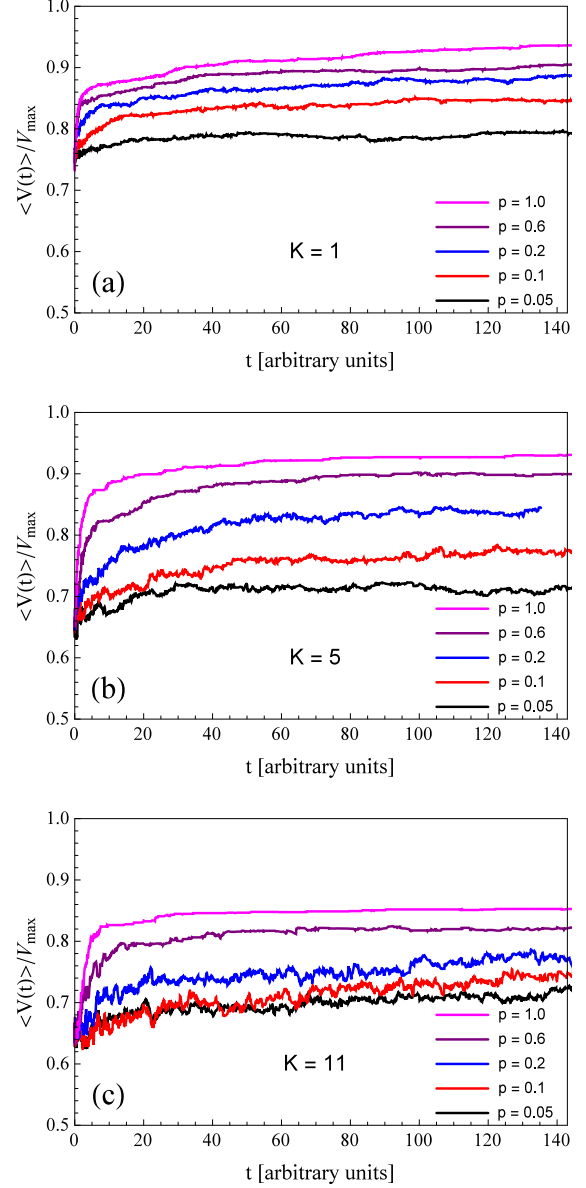


FIG. 6: The time-evolution of the normalized average group fitness  $\langle V(t) \rangle / V_{\max}$  for  $\beta J = 0.5$ ,  $p = 0.05, 0.1, 0.2, 0.6, 1.0$ .  $K = 1$ , (a);  $K = 5$ , (b);  $K = 11$ , (c).

highest group fitness value is obtained at the boundary between the increasing and almost stationary regions of  $\langle C_\infty \rangle$ . Moreover, results show that high consensus is necessary to guarantee high efficacy of the decision-making process, i.e. high values of  $\langle V_\infty \rangle / V_{\max}$ . This suggests that the decision making becomes optimal, i.e. the group as a whole is characterized by a higher degree of intelligence, at the point where the system dynamics changes qualitatively. This aspect of the problem is investigated in Figure 5 where the stationary values of normalized group-fitness and consensus are shown as a function of

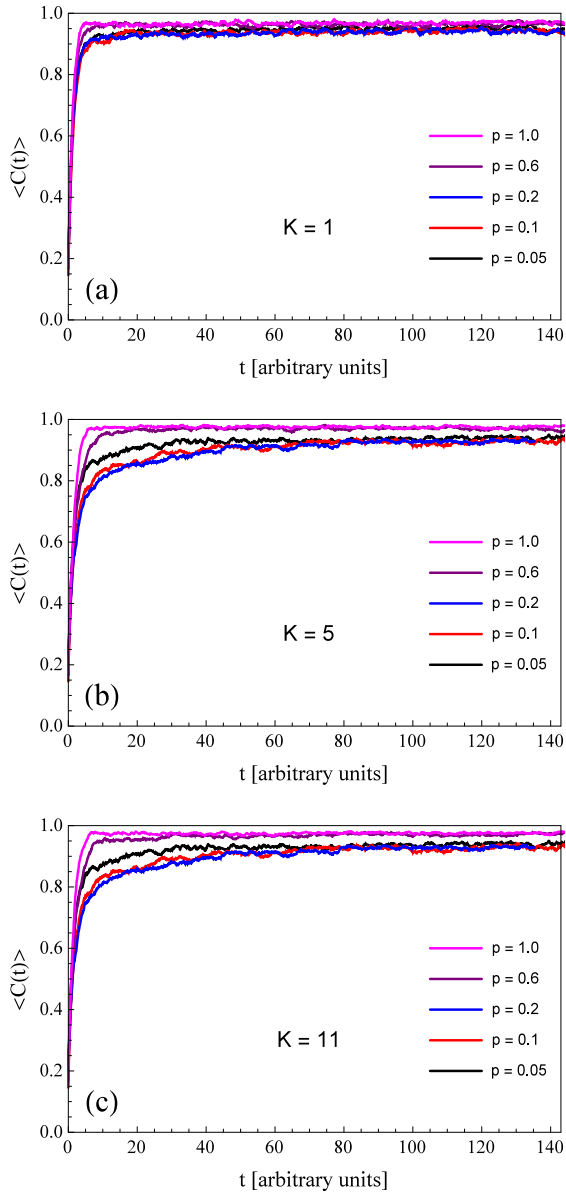


FIG. 7: The time-evolution of the statistically averaged consensus  $\langle C(t) \rangle$  for  $\beta J = 0.5$ ,  $p = 0.05, 0.1, 0.2, 0.6, 1$ .  $K = 1$ , (a);  $K = 5$ , (b);  $K = 11$ , (c).

the quantity  $\beta J M$  for different sizes  $M = 6, 12, 24$ , for  $N = 12$  and  $K = 5$ . Notably, the transition from low to high fitness values is always accompanied by an analogous transition from low to high consensus of the group. The transition becomes sharper and sharper as the group size  $M$  is incremented. In all cases the transition occurs for  $\beta J M \approx 1$ . This value is particularly interesting as it can be easily shown, by using a mean-field approach (see Appendix E), that for large Ising systems  $M \gg 1$  and in the case of complete graphs, the critical values  $(\beta J)_c$ , at which consensus sets in, satisfies the relation

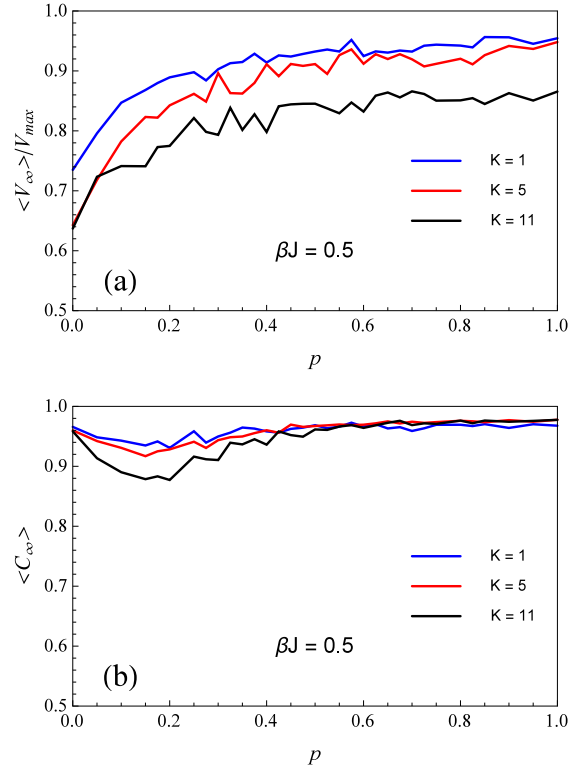


FIG. 8: The stationary values of the normalized average group fitness  $\langle V_\infty \rangle / V_{\max}$  as a function of  $p$ , (a); and of the statistically averaged consensus  $\langle C_\infty \rangle$  as a function of  $p$ , (b). Results are presented for  $\beta J = 0.5$ ,  $K = 1, 5, 11$

$(\beta J)_c = 1/M$ , in perfect agreement with our small-size numerical calculations. This is a very important result, which has analogies in many different self-organizing systems, as flocking systems and information flow processing [60–64].

In Figure 6 we investigate the influence of the level of knowledge  $p$  of members on the time-evolution of the normalized average fitness  $\langle V(t) \rangle / V_{\max}$ . Results are presented for  $\beta J = 0.5$ ,  $K = 1, 5, 11$ , and for different values of  $p = 0.05, 0.1, 0.2, 0.6, 1.0$ . Results show that improving the knowledge of the members, i.e. increasing  $p$ , enhances the performance of the decision-making process. In particular, a higher steady-state normalized fitness  $\langle V_\infty \rangle / V_{\max}$ , and a faster convergence toward the steady-state are observed. Note also, that, especially in the case of high complexity [Figure 6(b,c)], increasing  $p$  above 0.2 reduces the fluctuations of  $\langle V(t) \rangle$ , as a consequence of the higher agreement achieved among the members at higher level of knowledge. This is clear in Figure 7, where the time-evolution of the consensus  $\langle C(t) \rangle$  is shown for  $\beta J = 0.5$ ,  $K = 1, 5, 11$ , and  $p = 0.05, 0.1, 0.2, 0.6, 1.0$ . In Figure 8 the steady-state values of the normalized group fitness  $\langle V_\infty \rangle / V_{\max}$  [Fig. 8(a)], and social consensus  $\langle C_\infty \rangle$  [Fig. 8(b)] are shown as a function of  $p$ , for  $\beta J = 0.5$



and the three considered values of  $K = 1, 5, 11$ . Note that as  $p$  is increased from zero, the steady state value  $\langle V_\infty \rangle / V_{\max}$  initially grows fast [Fig. 8(a)]. In fact, because of social interactions, increasing the knowledge of each member also increases the knowledge of the group as a whole. But, above a certain threshold of  $p$  the increase of  $\langle V_\infty \rangle / V_{\max}$  is much less significant. This indicates that the knowledge of the group is subjected to a saturation effect. Therefore, a moderate level of knowledge is already enough to guarantee very good performance of decision-making process, higher knowledge levels being only needed to accelerate the convergence of the decision-making process. Figure 8(b) shows that for vanishing values of  $p$  the consensus  $\langle C_\infty \rangle$  takes high values, as each member's choice is driven only by consensus seeking. Increasing  $p$  initially causes a decrease of consensus, as the self-interest of each member leads to a certain level of disagreement. However, a further increment of  $p$  makes the members' knowledge overlap so that the self-interest of each member almost points in the same direction, resulting in a consensus increase.

## V. CONCLUSIONS

In this paper we developed a model of collective decision-making in human groups in presence of complex environment. The model described the time evolution of group choices in terms of a time-continuous Markov process, where the transition rates have been defined so as to capture the effect of the two main forces, which drive the change of opinion of the members of the group. These forces are the rational behavior which pushes each member to increase his/her self-interest, and the social interactions, which push the members to reach a common opinion. Our study provides contribution to the literature identifying under which circumstances collective decision making is more performing. We found that a moderate strength of social interactions allows for knowledge transfer among the members, leading to higher knowledge level of the group as a whole. This mechanism, coupled with the ability to explore the fitness landscape, strongly improves the performance of the decision-making process. In particular we found that the threshold value of the social interaction strength, at which the entire group behaves as unique entity characterized by a higher degree of intelligence, is just the critical threshold at which the consensus among the members sets in. This value can be also calculated for large systems through mean-field techniques and results to be in perfect agreement with our small system calculations. One can therefore estimate for any given social temperature  $\beta^{-1}$  and social interaction strength  $J$  the optimal number  $M_{\text{opt}} \approx (\beta J)^{-1}$  of team members leading to the emergence of a superior intelligence of the group.

We also found that increasing the level of knowledge of the members improves performance. However, above a certain threshold the knowledge of the group saturates,

i.e. the performance of the collective decision-making process becomes much less sensitive to the level of knowledge of each single member. Therefore, we can state that the collective decision-making is very high-performing already at moderate level of knowledge of the members, and that very high knowledge of all members only serves to accelerate the convergence of the decision-making process. Our results also showed that human groups with optimal levels of members' knowledge and strength of social interactions very well manage complex problems.

## Appendix A: The $N - K$ fitness landscape generation

In the  $N - K$  model a real valued fitness is assigned to each bit string  $\mathbf{d} = (d_1, d_2, \dots, d_N)$ , where  $d_i = \pm 1$ . This is done by first assigning a real valued contribution  $W_i$  to the  $i$ -th bit  $d_i$ , and then by defining the fitness function as  $V(\mathbf{d}) = N^{-1} \sum_{j=1}^N W_j (d_j, d_1^j, d_2^j, \dots, d_K^j)$ . Each contribution  $W_i$  depends not just on  $i$  and  $d_i$  but also on  $K$  ( $0 \leq K < N$ ) other bits. Now let us define the substring  $\mathbf{s}_i = (d_i, d_1^i, d_2^i, \dots, d_K^i)$ , by choosing at random, for each bit  $i$ ,  $K$  other bits. Each single contribution  $W_i(\mathbf{s}_i)$  is then a random function of  $2^{K+1}$  possible values of  $\mathbf{s}_i$ , and its value is drawn from a uniform distribution. Thus, a random table of contributions is generated independently for each  $i$ -th bit, thus allowing the calculation of the fitness function  $V(\mathbf{d})$ . The reader is referred to Refs. [35–37] for more details on the  $N - K$  complex landscapes.

## Appendix B: The Glauber dynamics on general graphs

Consider the Ising model on a general graph with adjacency matrix  $A_{ij}$ . The total energy of the system is given in Eq. (4). In steady state conditions the stationary distribution of the probability of the states  $P_0(\mathbf{s})$  is given by the Boltzmann distribution

$$P_0(\mathbf{s}) = \frac{\exp[-\beta E(\mathbf{s})]}{Z} \quad (\text{B1})$$

where  $Z = \sum_{\mathbf{l}} \exp[-\beta E(\mathbf{s}_{\mathbf{l}})]$  is the partition function of the system. The detailed balance condition then requires that

$$\frac{w(\mathbf{s}_{\mathbf{l}} \rightarrow \mathbf{s}'_{\mathbf{l}})}{w(\mathbf{s}'_{\mathbf{l}} \rightarrow \mathbf{s}_{\mathbf{l}})} = \frac{P_0(\mathbf{s}'_{\mathbf{l}})}{P_0(\mathbf{s}_{\mathbf{l}})} = \frac{\exp[-\beta E(\mathbf{s}'_{\mathbf{l}})]}{\exp[-\beta E(\mathbf{s}_{\mathbf{l}})]} \quad (\text{B2})$$

Now observe that

$$E(\mathbf{s}_{\mathbf{l}}) = -J s_{\mathbf{l}} \sum_j A_{lj} s_j - \frac{1}{2} J \sum_{ij \neq \mathbf{l}} A_{ij} s_i s_j \quad (\text{B3})$$



Substituting Eqs. (B3) in Eq. (B2) and recalling that  $\exp(x) = \cosh(x) + \sinh(x)$  we get

$$\frac{w(\mathbf{s}_l \rightarrow \mathbf{s}'_l)}{w(\mathbf{s}'_l \rightarrow \mathbf{s}_l)} = \frac{1 - \tanh\left(\beta J s_l \sum_j A_{lj} s_j\right)}{1 + \tanh\left(\beta J s_l \sum_j A_{lj} s_j\right)} \quad (\text{B4})$$

Noting that  $s_l = \pm 1$ , so that  $\tanh\left(\beta J s_l \sum_j A_{lj} s_j\right) = s_l \tanh\left(\beta J \sum_j A_{lj} s_j\right)$  we finally obtain

$$\frac{w(\mathbf{s}_l \rightarrow \mathbf{s}'_l)}{w(\mathbf{s}'_l \rightarrow \mathbf{s}_l)} = \frac{1 - s_l \tanh\left(\beta J \sum_j A_{lj} s_j\right)}{1 + s_l \tanh\left(\beta J \sum_j A_{lj} s_j\right)} \quad (\text{B5})$$

Therefore, a possible choice for the transition rates for the Ising-Glauber dynamics on general graph is

$$w(\mathbf{s}_l \rightarrow \mathbf{s}'_l) = \alpha \left[ 1 - s_l \tanh\left(\beta J \sum_j A_{lj} s_j\right) \right] \quad (\text{B6})$$

where  $\alpha$  is an arbitrary constant. We have chosen  $\alpha = 1/2$ .

### Appendix C: Detailed balance condition

Here we show that the transition rate given in Eq. (6) fulfils the detailed balance condition of Markov chains, which requires the existence of a stationary probability distribution  $P_0(\mathbf{s}_l)$  such that

$$\frac{P_0(\mathbf{s}'_l)}{P_0(\mathbf{s}_l)} = \frac{w(\mathbf{s}_l \rightarrow \mathbf{s}'_l)}{w(\mathbf{s}'_l \rightarrow \mathbf{s}_l)} \quad (\text{C1})$$

Using Eq. (6) the above condition Eq. (C1) writes

$$\begin{aligned} \frac{P_0(\mathbf{s}'_l)}{P_0(\mathbf{s}_l)} &= \frac{1 - s_l \tanh(\beta J \sum_h A_{lh} s_h)}{1 + s_l \tanh(\beta J \sum_h A_{lh} s_h)} \\ &\times \frac{\exp\{\beta' [\bar{V}(\mathbf{s}'_l) - \bar{V}(\mathbf{s}_l)]\}}{\exp\{\beta' [\bar{V}(\mathbf{s}_l) - \bar{V}(\mathbf{s}'_l)]\}} \end{aligned} \quad (\text{C2})$$

and recalling Eqs. (B2, B5) yields

$$\frac{P_0(\mathbf{s}'_l)}{P_0(\mathbf{s}_l)} = \frac{\exp[-\beta E(\mathbf{s}'_l) + 2\beta' \bar{V}(\mathbf{s}'_l)]}{\exp[-\beta E(\mathbf{s}_l) + 2\beta' \bar{V}(\mathbf{s}_l)]} \quad (\text{C3})$$

this allows to define the stationary probability distribution

$$P_0(\mathbf{s}_l) = \frac{\exp[-\beta E(\mathbf{s}_l) + 2\beta' \bar{V}(\mathbf{s}_l)]}{\sum_k \exp[-\beta E(\mathbf{s}_k) + 2\beta' \bar{V}(\mathbf{s}_k)]} \quad (\text{C4})$$

which satisfies the detailed balance condition Eq. (C1).

### Appendix D: The stochastic simulation algorithm

The stochastic simulation algorithm we use to solve the Markov process (5) is derived from the one proposed by Gillespie [58, 59]. We just summarize the main steps of the algorithm:

1. Choose a random initial state  $\mathbf{s}$  of the system
2. Calculate the transition rates  $w(\mathbf{s}_l \rightarrow \mathbf{s}'_l)$
3. Calculate the total rate  $w_T = \sum_l w(\mathbf{s}_l \rightarrow \mathbf{s}'_l)$
4. Normalize the transition rates as  $\nu_l = w(\mathbf{s}_l \rightarrow \mathbf{s}'_l) / w_T$
5. Construct the cumulative distribution  $F(\nu_l)$  from the probability mass function  $\nu_l$
6. Calculate the time  $\Delta t$  to the next opinion flip drawing from an exponential distribution with mean  $1/w_T$ , i.e. choose a real random number  $0 \leq r < 1$  from a uniform distribution and set  $\Delta t = -w_T^{-1} \log(r)$ .
7. Identify the  $k$ -th opinion  $s_k$  which flips from  $s_k$  to  $-s_k$  by drawing from a discrete distribution with probability mass function  $\nu_l$ , i.e. draw a real random number  $0 \leq s < 1$  from a uniform distribution and choose  $k$  so that  $F(\nu_{k-1}) \leq s < F(\nu_k)$ .
8. Update the state vector and return to step 2 or quit.

### Appendix E: The mean-field calculations of the Ising model on a complete graph.

On a complete graph the total energy of a system of  $M$  spins is

$$E = - \sum_{k < h} J \sigma_k \sigma_h \quad (\text{E1})$$

and the average magnetization is  $\langle \sigma \rangle = M^{-1} \sum_k \langle \sigma_k \rangle$ . Using Eq. (B6) the Ising-Glauber rate becomes

$$w_k = w(\sigma_k \rightarrow -\sigma_k) = \frac{1}{2} \left[ 1 - \sigma_k \tanh\left(\beta J \sum_{j \neq k} \sigma_j\right) \right] \quad (\text{E2})$$

Using Eq. (5) one can easily derive the following equation of motion for the average magnetization  $\langle \sigma_k \rangle$  of the  $k$ -th site

$$\frac{d\langle \sigma_k \rangle}{dt} = -2 \langle w_k \sigma_k \rangle = -\langle \sigma_k \rangle + \left\langle \tanh\left(\beta J \sum_j \sigma_j\right) \right\rangle \quad (\text{E3})$$

Assuming that  $M$  is large, using  $\langle \sigma \rangle = M^{-1} \sum_k \langle \sigma_k \rangle$ , and exploiting the mean field approach we write  $\langle \tanh [\beta J \sum_j \sigma_j] \rangle = \tanh [\beta J \sum_j \langle \sigma_j \rangle] = \tanh [\langle \sigma \rangle \beta J M]$ , and

$$\frac{d\langle \sigma \rangle}{dt} = -\langle \sigma \rangle + \tanh [\langle \sigma \rangle \beta J M] \quad (\text{E4})$$

The average magnetization  $\langle \sigma \rangle$  at the fixed point of Eq. (E4) satisfies the relation

$$\langle \sigma \rangle = \tanh [\langle \sigma \rangle M \beta J] \quad (\text{E5})$$

For  $M\beta J \leq 1$  only the trivial solution  $\langle \sigma \rangle_1 = 0$  can be found. However, for  $M\beta J > 1$  other two solutions

$\langle \sigma \rangle_2 = -\langle \sigma \rangle_3 > 0$  appear which depends on the specific value of  $M\beta J$ . In this case  $\langle \sigma \rangle_1 = 0$  becomes unstable. Thus, the critical point for the phase transition is

$$(\beta J)_c = \frac{1}{M} \quad (\text{E6})$$

The above equation is in perfect agreement with the results shown in Fig. 5, thus confirming that the decision making process of the group becomes optimal just when the system changes qualitatively its dynamics.

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